Nonlinear Electromagnetism in General Relativity

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March 2014

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Abstract

This project studies infinitesimal perturbations of the spherically symmetric solution to the Einstein-Maxwell system of PDEs with nonlinear Born-Infeld Lagrangian. The motivation comes from the search for a consistent theory of electromagnetism that can describe the spacetime of a single point charge. Making use of nonlinear electromagnetic theories has helped to avoid difficulties such as infinite self-energy and strong naked singularities, but up until now most work has focused on finding spherically symmetric solutions. Since real point charges such as electrons and protons possess an intrinsic angular momentum ("spin"), it is important to investigate how removal of spherical symmetry affects the spacetime, as this will enable us to assess the validity of the Born-Infeld Lagrangian in a description of real charged particles. In this project, a linearisation of the perturbed Einstein-Maxwell system was found, but the length and complexity of the perturbed equations made detailed analysis of the resulting system difficult.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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1 Introduction

The motivation for this project comes from the search for a consistent theory of electromagnetism that can describe the spacetime of a single point charge. In linear Maxwell theory several difficulties arise, including the infinite self-energy of the point charge and the presence of a strong naked singularity (i.e. one which is not covered by an event horizon) on the time axis. As a solution to these problems, Max Born proposed that the equations for electrodynamics be made nonlinear.[4] Some work has been done on trying to understand the properties of spherically symmetric nonlinear electromagnetic theories[12], but up until now no attempts have been made to examine the effects of removing spherical symmetry. In this project, an infinitesimal perturbation to the spherically symmetric solution is studied as a first step to investigating how the loss of spherical symmetry affects the spacetime.

The Einstein-Maxwell system of PDEs can be derived by coupling Einstein's field equations to the Maxwell equations for electromagnetism, and reads

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} & \text{(1.1a)} \\ dF = 0 & \text{(1.1b)} \\ dM = 0 & \text{(1.1c)} \end{cases}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the stress-energy tensor, F is the Faraday tensor of the electromagnetic field, and M is the Maxwell tensor corresponding to F (each of which is defined in Section 3). M and F are related by an "aether law" which can be derived from the Lagrangian of the system. In linear Maxwell theory, this law is given by

$$M = - * F \tag{1.2}$$

where $*$ is the Hodge star operator. The unique spherically symmetric asymptotically flat solution of this system is known as the Reissner-Weyl-Nordström spacetime (RWN), and is described in Section 4.1.1.

There are several difficulties associated with the RWN solution, the first being that the self-energy of the point charge is infinite. Since mass and energy are equivalent in general relativity, this means that the total electrostatic energy of a nonzero charge makes an infinite contribution to the total mass of the spacetime. A further difficulty with the RWN solution is the presence of a strong eternal naked singularity at the location of the charge. Because the singularity is eternal, this spacetime cannot arise as a solution of a classically posed initial value problem; furthermore the stability of such a singularity is as yet unknown. Several attempts have been made to solve these problems, but none have been completely successful so far.

Choosing a nonlinear electromagnetic Lagrangian density can solve the first of these two problems, and can also manage to reduce the strength of the spacetime singularity. One example of such a Lagrangian density is the one-parameter family proposed by Born and Infeld[5]:

$$L_\beta = *\frac{1}{4\pi\beta^4}\left[1 - \sqrt{1 - \beta^4 * (F \wedge *F) - \beta^8(*(F \wedge F))^2}\right] \tag{1.3}$$

where $\beta > 0$ is a parameter that gives a limit to the size of the field. L_β also has the property that it reduces to the Lagrangian of the Maxwell-Maxwell system in the weak field limit.

A. Shadi Tahvildar-Zadeh's paper *On the Static Spacetime of a Single Point Charge*[12] studies spherically symmetric, asymptotically flat, electrostatic solutions to nonlinear electromagnetic theories. It is shown that when certain conditions are satisfied by the aether law, there exists a solution of the Einstein-Maxwell system with that aether law, unique in the spherically symmetric class, with the mildest possible singularity (a conical singularity[1] on the time axis). However it is known that in reality, particles such as electrons are not spherically symmetric - they possess an intrinsic angular momentum. In this project, an infinitesimal perturbation is added to this spherically symmetric solution. This perturbed quantity is reinserted into the Einstein-Maxwell system (keeping only first-order terms), and the resulting linearised equations are investigated. There are three possible outcomes of this calculation:

(i) $\exists$ a solution for each mass, charge and infinitesimal angular momentum, as was the case in linear Maxwell theory.

(ii) $\exists$ a solution for some values of infinitesimal angular momentum but not for others, meaning that the parameters mass, charge, angular momentum and β would have to satisfy some constraint.

(iii) $\nexists$ solutions for nonzero infinitesimal angular momentum; or else $\exists$ solutions, but with problems of infinite energy and/or a strong singularity. This would indicate that the Born-Infeld Lagrangian may not be valid as a description of the real electron.

It has not yet been determined which of these possibilities holds. Several difficulties have been encountered, most relating to the greatly increased length and complexity of

[1] A conical singularity is a point at which the limit of all covariant quantities are finite, but spacetime is not smooth.

the equations on addition of an infinitesimal perturbation. Because of this, alternative methods for carrying out the calculation are currently being considered. One possibility would be to return to the Lagrangian of the system and to use variational principles to derive the linearised equations directly. It is hoped that the resulting equations will be easier to work with than those that have been found using the original method.

The remainder of this work is organised as follows: In **Section 2**, the essential mathematical concepts from differential geometry and tensor calculus will be introduced. **Section 3** will discuss the Einstein-Maxwell system of partial differential equations, which describes the geometry of a spacetime endowed with an electromagnetic field. **Section 4** will give an overview of the difficulties associated with point defects in electromagnetic spacetime theories, and will introduce the concept of nonlinear electromagnetism as a possible solution to these difficulties, in particular for a spherically symmetric spacetime. **Section 5** will discuss an attempt to remove this spherical symmetry through an infinitesimal perturbation.

2 Preliminary Mathematics

The purpose of this section is to introduce the essential mathematical concepts that were needed for this project. Much of what follows is adapted from G.F.R. Ellis's *Course on General Relativity.*[7]

2.1 Spacetime

Spacetime is a four-dimensional manifold $\mathcal{M}$ (i.e. a topological space that is locally like Cartesian space), whose points give the entire set of "events" in space and time where objects can exist. Each event in spacetime can be described using four coordinates,

$$\{x^i\} = (x^0, x^1, x^2, x^3)$$

which can be chosen arbitrarily. In general several overlapping coordinate systems are needed to describe the whole of spacetime.

2.1.1 Tangent Vectors, Covectors and Differentials

A curve in spacetime is a local map of $\mathbb{R}^1$ into $\mathcal{M}$, i.e. it is a one-dimensional set of events in the spacetime, for example the history of a point particle in spacetime or the path taken by a light-ray. The curve is denoted by $x^\alpha(\tau)$ where τ is the curve parameter. Each curve is associated with a tangent vector that gives its direction in spacetime. This vector has components

$$X^\alpha = \frac{dx^\alpha}{d\tau} \tag{2.1}$$

If P is a point on $\mathcal{M}$, then the tangent space $T_P\mathcal{M}$ is the set of vectors that are tangent to $\mathcal{M}$ at P. Components of tangent vectors are generally written with upper indices.

Given any vector space, the dual space is the set of all linear maps from the vector space to $\mathbb{R}$. The dual space of $T_P\mathcal{M}$ is denoted $T^*_P\mathcal{M}$, and is known as the co-tangent space. Elements of $T^*_P\mathcal{M}$ are known as covectors (or one-forms), and their components are written using lower indices to distinguish them from vectors.

If $f : \mathcal{M} \to \mathbb{R}$ is a smooth function on the spacetime, its differential $df \in T^*_P\mathcal{M}$ at a point $P \in \mathcal{M}$ is defined by

$$df(X_P) - X_P f \tag{2.2}$$

where X_P is any vector in the tangent space $T_P\mathcal{M}$ at P. The differential gives the variation of the function across its level surfaces (i.e. surfaces of constant f), and has components

$$df_\alpha = \frac{\partial f}{\partial x^\alpha} \tag{2.3}$$

2.1.2 The Metric Tensor

Because space and time are merged together, the concepts of distance and time intervals are not well-defined - they depend on the particular coordinate system being used. However, one can define a spacetime interval that gives a coordinate-independent concept of "distance" on the spacetime manifold. The spacetime interval corresponding to a small coordinate displacement $dx^\mu = (dx^0, dx^1, dx^2, dx^3)$ is defined as [2]

$$ds^2 = g_{\mu\nu}(x^\alpha)dx^\mu dx^\nu \tag{2.4}$$

where $g_{\mu\nu}(x^\alpha)$ are components of the metric tensor. As an example, the metric tensor for flat spacetime in polar coordinates $\{x^\alpha\} = (t, r, \theta, \phi)$ can be written $g_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2 \sin^2\theta)$, giving the spacetime interval

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

The signature of a metric is defined as the pair of integers (p, r) where p is the number of positive eigenvalues of the metric and r is the number of negative eigenvalues. Spacetime can be modelled as a 4-dimensional Lorentzian manifold, which is one with signature $(3, 1)$.

2.1.3 Static and Stationary Manifolds

A Killing field on a manifold is a vector field that preserves the metric.[3] A Lorentzian manifold $\mathcal{M}$ is called stationary if it has a Killing field whose orbits are complete and everywhere timelike. It is called static if this Killing field is hypersurface-orthogonal everywhere.

2.2 Tensors

When working in spacetime, the equations of physics should be valid regardless of the coordinate system chosen. We must work with quantities that are invariant under a change of coordinates (although their components may not be).

Suppose we change from coordinates $\{x^\alpha\}$ to new coordinates $\{x^{\alpha'}\}$, which are given by

$$x^{\alpha'} = x^{\alpha'}(x^\alpha) \tag{2.5}$$

[2]Here we use Einstein's Summation Convention, namely that whenever an index is repeated, it is summed over the range of the index. So we have $X^\alpha X_\alpha = X^0X_0 + X^1X_1 + X^2X_2 + X^3X_3$.

[3]More precisely, a vector field X is a Killing field if the Lie derivative (see Section 2.2.4) with respect to X of the metric g vanishes: $\mathcal{L}_X g = 0$.

It is easy to show that the components of a vector $X^\alpha \in T_P\mathcal{M}$ (one upper index) in this new coordinate system are given by

$$X^{\alpha'} = A^{\alpha'}{}_{\alpha} X^\alpha \tag{2.6}$$

where

$$A^{\alpha'}{}_{\alpha} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \tag{2.7}$$

Similarly, it can be shown that the components of a covector $\omega \in T_P^*\mathcal{M}$ (one lower index) transform as

$$\omega' = A_{\alpha'}{}^{\alpha} \omega \tag{2.8}$$

where $A_{\alpha'}{}^{\alpha}$ is the inverse of $A^{\alpha'}{}_{\alpha}$. In short, any quantity written with a single upper index can be transformed using Equation 2.6, while any quantity written with a single lower index can be transformed using Equation 2.8.

A tensor is a quantity that generalises these patterns for any number of upper and lower indices, transforming each upper index in the same way as a vector and each lower index in the same way as a differential. Thus a general tensor $T^{\mu_1,\ldots\mu_r}{}_{\nu_1,\ldots\nu_s}$ transforms as follows:

$$T^{\mu'_1\ldots\mu'_r}{}_{\nu'_1\ldots\nu'_s} = A^{\mu'_1}{}_{\mu_1} \cdots A^{\mu'_r}{}_{\mu_r} A_{\nu'_1}{}^{\nu_1} \cdots A_{\nu'_s}{}^{\nu_s} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} \tag{2.9}$$

A tensor with r upper indices and s lower indices is called an $\binom{r}{s}$ tensor.

The fundamental point of tensor transformations is that if a tensor equation is true in one coordinate system, then it is true in all coordinate systems.

2.2.1 Algebraic Operations on Tensors

There are four basic algebraic operations on tensors:

(i) *Linear Combination*

Given two tensors of the same type, we may form a new tensor by taking a linear combination. For example, if $R^\mu{}_\nu$ and $S^\mu{}_\nu$ are tensors and $\alpha, \beta \in \mathbb{R}$, then

$$T^\mu{}_\nu = \alpha R^\mu{}_\nu + \beta S^\mu{}_\nu$$

is also a tensor.

(ii) *Multiplication*

Given any two tensors, they can be multiplied to form a new tensor. For example, if $R^\mu{}_\nu$ and $S^{\kappa\lambda}$ are tensors, then

$$T^\mu{}_\nu{}^{\kappa\lambda} \equiv (R \otimes S)^\mu{}_\nu{}^{\kappa\lambda} = R^\mu{}_\nu S^{\kappa\lambda}$$

is also a tensor.

(iii) *Contraction*

Given a tensor with at least one upper index and at least one lower index, we can form a new tensor by setting one upper and one lower index equal to one another and summing over all possible values of the index. For example, if $S_\mu{}^\nu{}_{\kappa\lambda}$ is a tensor then

$$R_{\mu\lambda} = S_\mu{}^\nu{}_{\nu\lambda}$$

is also a tensor.

(iv) *Raising and Lowering Indices*

To lower indices we contract with the metric tensor. For example,

$$X_\alpha = g_{\alpha\beta} X^\beta$$

To raise indices we contract with the inverse metric tensor $g^{\alpha\beta}$. For example,

$$X^\alpha = g^{\alpha\beta} X_\beta$$

2.2.2 Covariant Differentiation

In order to set up differential equations for tensors that are valid in an arbitrary coordinate system, one needs to define covariant differentiation of tensors. Differentiation of tensors is problematic, because taking an ordinary partial derivative leads to a quantity that is not in fact a tensor.

Instead, given two vectors X and Y, we define a map $\nabla Y : X \to \nabla_X Y$, the covariant derivative of Y along X. The covariant derivative has the following properties:

(i) *Leibniz Rule*

The standard Leibniz differentiation rule $(D(fg) = D(f)g + fD(g))$ applies to covariant differentiation, i.e. for a scalar field f and vector fields X and Y,

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y \tag{2.10}$$

where we have defined $\nabla_X f = X(f)$ for scalar fields f.

(ii) *Linearity in lower argument*

If f,g are scalar fields and X,Y,Z are vector fields then we have

$$\nabla_{fX+gZ}(Y) = f\nabla_X(Y) + g\nabla_Z(Y) \tag{2.11}$$

(iii) *Commutes with addition of vectors in the upper argument*

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z \tag{2.12}$$

In an arbitrary basis $\{e_\mu\}$, the covariant derivative ∇_{e_ν} maps each basis vector e_μ to a vector field $\nabla_{e_\nu} e_\mu \equiv \nabla_\nu e_\mu$, which can be written as a linear combination of basis vectors:

$$\nabla_\nu e_\mu = \Gamma^\lambda{}_{\mu\nu} \tag{2.13}$$

where the $\Gamma^\lambda{}_{\mu\nu}$ are known as connection coefficients (or Christoffel symbols in a coordinate basis). In a coordinate basis, the covariant derivative of a vector X^μ has components

$$X^\mu{}_{;\nu} = X^\mu{}_{,\nu} + \Gamma^\mu{}_{\nu\lambda} X^\lambda \tag{2.14}$$

where ";" denotes covariant differentiation and "," denotes partial differentiation. The Christoffel symbols can be defined in terms of the metric using the following relation:

$$\Gamma^\mu{}_{\nu\lambda} = g^{\mu\kappa}(g_{\kappa\lambda,\nu} + g_{\nu\kappa,\lambda} - g_{\lambda\nu,\kappa}) \tag{2.15}$$

To extend the idea of covariant differentiation to arbitrary tensors, one can make use of the properties of the covariant derivative mentioned above, along with the following further property:

(iv) *Commutes with contraction*

For example, if $R_{\mu\lambda} = S_\mu{}^\nu{}_{\nu\lambda}$ then $R_{\mu\lambda;\kappa} = S_\mu{}^\nu{}_{\nu\lambda;\kappa}$.

Using these properties, one finds the general rule for covariant differentiation of tensors in a coordinate basis:

$$\begin{aligned} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s;\lambda} = T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s,\lambda} &+ \Gamma^{\mu_1}{}_{\sigma\lambda} T^{\sigma\mu_2\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \cdots + \Gamma^{\mu_r}{}_{\sigma\lambda} T^{\mu_1\ldots\sigma}{}_{\nu_1\ldots\nu_s} \\ &- \Gamma^\sigma{}_{\nu_1\lambda} T^{\mu_1\ldots\mu_r}{}_{\sigma\nu_2\ldots\nu_s} - \cdots - \Gamma^\sigma{}_{\nu_s\lambda} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\sigma} \end{aligned} \tag{2.16}$$

2.2.3 Push-forward and Pull-back

If $\mathcal{M}$ and $\mathcal{N}$ are manifolds then a map $h : \mathcal{M} \to \mathcal{N}$ is said to be smooth if for every function $f : \mathcal{N} \to \mathbb{R}$ the function $f \circ h : \mathcal{M} \to \mathbb{R}$ is smooth. If h is smooth, then it maps a smooth curve γ in $\mathcal{M}$ to a smooth curve $h \circ \gamma$ in $\mathcal{N}$.

If $X \in T_P\mathcal{M}$ is a tangent vector to γ at a point P, then there exists a map

$$h_* : T_P\mathcal{M} \to T_{h(P)}\mathcal{N}$$

known as the push-forward to $h \circ \gamma$, which maps tangent vectors at $P \in \mathcal{M}$ to tangent vectors at $h(P) \in \mathcal{N}$. If $f : \mathcal{N} \to \mathbb{R}$ is a smooth function, then

$$(h_* X_P)(f) = X_P(f \circ h) \tag{2.17}$$

Analogously, there is a map between covectors

$$h^* : T^*_{h(P)}\mathcal{N} \to T^*_P\mathcal{M}$$

known as the pull-back. If $\omega \in T^*_{h(P)}\mathcal{N}$ and $X_P \in T_P\mathcal{M}$, then

$$(h^*\omega)(X_P) = \omega(h_* X_P) \tag{2.18}$$

If h is a diffeomorphism (i.e. differentiable and invertible with differentiable inverse), then it is possible to generalise the definition of the pull-back to a vector (i.e. a tensor with a single upper index). This can be very loosely defined as the push-forward of the vector under the inverse map. Combining this with the definition of the pull-back of a covector, one can generalise the pull-back as an operator on tensors with arbitrary indices.

2.2.4 The Lie Derivative

The Lie derivative is the tensor derivative associated with the process of dragging along. If a tensor T is dragged along a one-parameter diffeomorphism $h(\tau)$ with infinitesimal generator X, then the Lie derivative $\mathcal{L}_X$ at a point P gives the rate of change of the tensor in the direction of the curve. Mathematically, it is defined by

$$\mathcal{L}_X T = \frac{\partial}{\partial \tau}((h_\tau)^* T)_P \Big|_{\tau=0} \tag{2.19}$$

where $h_{\tau=0} = P$ and ξ^* is the pull-back operator.

In a coordinate basis, if $X^\alpha = \frac{\partial x^\alpha}{\partial \tau}$ is tangent vector to a curve $h(\tau) = x^\alpha(\tau)$ through

P, then the coordinates of $\mathcal{L}_X T$ are given by:

$$(\mathcal{L}_X T)^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} = T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s, \lambda} X^\lambda - T^{\lambda \mu_2 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} X^{\mu_1}{}_{,\lambda} - \cdots - T^{\mu_1 \ldots \lambda}{}_{\nu_1 \ldots \nu_s} X^{\mu_r}{}_{,\lambda}$$
$$+ T^{\mu_1 \ldots \mu_r}{}_{\lambda \nu_2 \ldots \nu_s} X^\lambda{}_{,\nu_1} + \cdots + T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \lambda} X^\lambda{}_{,\nu_s} \quad (2.20)$$

2.3 Differential Forms

A p-form is a $\binom{0}{p}$ tensor that is completely antisymmetric. Given a p-form P and a q-form Q we can form a $(p+q)$-form $P \wedge Q$ defined by

$$P \wedge Q = P \otimes Q - Q \otimes P \quad (2.21)$$

In a coordinate basis, a p-form P can be written

$$P = \frac{1}{p!} P_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \quad (2.22)$$

where we have $P_{\mu_1 \ldots \mu_p} = P_{[\mu_1 \ldots \mu_p]}$. Then the components of $P \wedge Q$ are given by

$$(P \wedge Q)_{\nu_1 \nu_2 \ldots \nu_{p+q}} = \frac{(p+q)!}{p! q!} P_{[\nu_1 \ldots \nu_p} Q_{\nu_{p+1} \ldots \nu_{p+q}]} \quad (2.23)$$

Two important operations on differential forms are the exterior derivative and the Hodge star operator, both of which are essential for the coordinate-independent formulation of Maxwell's equations.

2.3.1 Exterior Derivative

The exterior derivative is a map "d", which maps p-forms to $(p+1)$-forms, such that

$$d(P_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \sum_{i=1}^{n} \frac{\partial P_{\mu_1 \ldots \mu_p}}{\partial x^i} dx^i \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \quad (2.24)$$

If A is a p-form such that $dA = 0$, we say A is closed. If there exists a $(p-1)$-form B such that $A = dB$ then we say that A is exact.

2.3.2 Hodge Star Operator

If $\sigma_{\mu_1 \ldots \mu_p}$ is a p-form defined in four-dimensional spacetime then the Hodge star operator acts on σ to give a $(4-p)$-form $*\sigma$ defined by

$$*\sigma_{\mu_1 \ldots \mu_{4-p}} = \frac{1}{p!} \sigma^{\nu_1 \ldots \nu_p} \eta_{\nu_1 \ldots \nu_p \mu_1 \ldots \mu_{4-p}} \quad (2.25)$$

where η is the volume form, given by

$$\eta_{\alpha\beta\gamma\delta} = -\sqrt{|g|}\epsilon_{\alpha\beta\gamma\delta} \tag{2.26}$$

where $g = \det(g_{\mu\nu})$ and $\epsilon_{\alpha\beta\gamma\delta}$ is the totally anti-symmetric tensor:

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{if}\,(\alpha,\beta,\gamma,\delta)\,\text{is an even permutation of}\,(0,1,2,3) \\ -1 & \text{if}\,(\alpha,\beta,\gamma,\delta)\,\text{is an odd permutation of}\,(0,1,2,3) \\ 0 & \text{otherwise} \end{cases} \tag{2.27}$$

3 The Einstein-Maxwell System

Here Einstein's field equations for general relativity and the equations of electrodynamics will be introduced. They will then be combined to form the Einstein-Maxwell system of PDEs, which describes the geometry of a spacetime endowed with an electromagnetic field.

3.1 Curvature and Einstein's Field Equations

In order to distinguish between flat and curved spacetimes, one needs the concept of curvature. There is no simple criterion for flatness based on the metric tensor itself, as there are infinitely many coordinate systems to choose from, each of which produces a different form for the metric.

3.1.1 The Riemann Curvature Tensor

The Riemann curvature tensor field is a $\binom{1}{3}$ tensor field defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{XY-YX} Z \tag{3.1}$$

where X,Y,Z are vector fields. In a coordinate basis, its components are given by

$$R_{\mu}{}^{\nu}{}_{\lambda\rho} := \Gamma^{\nu}{}_{\lambda\mu,\rho} - \Gamma^{\nu}{}_{\rho\mu,\lambda} + \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\nu}{}_{\rho\sigma} - \Gamma^{\sigma}{}_{\rho\mu}\Gamma^{\nu}{}_{\lambda\sigma} \tag{3.2}$$

The Riemann curvature tensor contains all the information on space-time curvature. In particular, it vanishes if and only if spacetime is locally flat.

The curvature tensor has four important symmetries:

(i) It is skew-symmetric in the last pair of indices:

$$R_{\mu}{}^{\nu}{}_{\lambda\rho} = R_{\mu}{}^{\nu}{}_{[\lambda\rho]} \tag{3.3}$$

(ii) Using the metric tensor to lower the second index, it is also skew in the first pair of indices:

$$R_{\mu\nu\lambda\rho} = R_{[\mu\nu]\lambda\rho} \tag{3.4}$$

(iii) It obeys a cyclic identity:

$$R_{\mu\nu\lambda\rho} = R_{\mu[\nu\lambda\rho]} \tag{3.5}$$

(iv) It is symmetric under interchange of the pairs of skew indices:

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu} \tag{3.6}$$

3.1.2 Contractions of the Curvature Tensor

Because of the above symmetries, there is only one non-trivial contraction of the Riemann tensor:

$$R_{\mu\nu} := R_{\mu}{}^{\lambda}{}_{\nu\lambda} \tag{3.7}$$

This is called the Ricci tensor.
The Ricci scalar is formed by contracting the Ricci tensor:

$$R := R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{3.8}$$

One final contraction of the curvature tensor is the Kretschmann scalar

$$\mathcal{K} := R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \tag{3.9}$$

The Kretschmann scalar can be used to distinguish between coordinate singularities and "real" singularities. Since $\mathcal{K}$ is a scalar, it is coordinate independent, so if a singularity appears in the expression for $\mathcal{K}$ then this singularity is an intrinsic property of the spacetime.

3.1.3 Einstein's Field Equations

In general relativity, gravity is treated geometrically, as due to spacetime curvature. Einstein's basic idea was that matter determines the geometry of spacetime.

There is one tensor that is characteristic of all types of matter, the stress-energy tensor $T_{\mu\nu}$. Hence it is this tensor that is used in the equations that determine the spacetime structure, through Einstein's field equations:[4]

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.10}$$

where κ is a coupling constant and $G_{\mu\nu}$ is Einstein's tensor, defined as follows:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{3.11}$$

[4] More generally, these equations are $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$ where Λ is the cosmological constant. Here we will set $\Lambda = 0$.

So we have

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.12}$$

The constant κ can be obtained by comparison with the Newtonian limit and is found to be equal to $8\pi G$, where G is Newton's gravitational constant.

Multiplying Einstein's field equations by $g^{\mu\nu}$ and contracting gives

$$R - \frac{1}{2}R(4) = \kappa T \tag{3.13}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the stress-energy tensor. Rearranging, one finds that

$$R = -\kappa T \tag{3.14}$$

and substituting back into Equation 3.12 gives an alternate form of the Einstein Field Equations:

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{3.15}$$

3.1.4 ADM Mass

In general relativity, there is no local quantity that behaves like classical energy, since there is no good local definition of mass and energy. This is clear if one considers the following. Intuition tells us that

(i) Mass is additive.

(ii) Energy and mass are related by $E = mc^2$.

(iii) A stable configuration is achieved when the total energy of the system is $\leq$ the energies of the individual components.

Clearly, these three statements contradict each other, so the classical definitions of mass and energy cannot be used consistently.

However, it is possible to introduce a global definition. For a spacetime that is asymptotically flat (i.e. approaches Minkowski space at infinity), we can define a total mass, known as the ADM mass, which essentially gives the strength of the gravitational field at infinity.

3.2 Equations for Electrodynamics

The equations for electrodynamics can be formulated in a coordinate-independent way using differential forms. Here we will introduce the concepts for a general nonlinear theory.

They can then be easily specialised to the linear Maxwell case by choosing the appropriate Lagrangian. Much of what follows is adapted from A. Shadi Tahvildar-Zadeh's paper, *On the Static Spacetime of a Single Point Charge.*[12]

3.2.1 Source-Free Maxwell Equations

Suppose $(\mathcal{M}, g)$ is a 4-dimensional Lorentzian manifold. If a is a 1-form, f is a 2-form and $L_{em}(a, f)$ is a 4-form on $\mathcal{M}$, then the electromagnetic action is by definition

$$\mathcal{S}[a] := \int L_{em}(a, da) \tag{3.16}$$

A critical point of $\mathcal{S}$ with respect to variations of a is called an electromagnetic potential A, and its exterior derivative is the electromagnetic Faraday tensor $F = dA$. By definition, the Maxwell tensor M is

$$M = \left.\frac{\partial L_{em}}{\partial f}\right|_{a=A, f=F} \tag{3.17}$$

The source-free Maxwell equations are the Euler-Lagrange equations for stationary points of $\mathcal{S}$, and are equivalent to the system

$$\begin{cases} \text{dF=0} & (3.18\text{a}) \\ \text{dM=0} & (3.18\text{b}) \end{cases}$$

It can be shown that in four-dimensional spacetime, every Lorentz-invariant gauge-invariant source-free electromagnetic Lagrangian L_{em} can be written in the form

$$L_{em}(a, f) = -l(x(f), y(f))\epsilon[g] \tag{3.19}$$

where x and y are the electromagnetic invariants:

$$x(f) := -\frac{1}{2} * (f \wedge *f) = \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \tag{3.20}$$

$$y(f) := \frac{1}{2} * (f \wedge f) = \frac{1}{4} f_{\mu\nu} * f^{\mu\nu} \tag{3.21}$$

It is clear from Equations 3.17 and 3.19 that $*L_{em} = l$ and

$$*M = \frac{\partial l}{\partial F} = l_x F + l_y * F \tag{3.22}$$

This relation between M and F is known as an "aether law". For linear Maxwell theory, we have $l = x$ and $M = - * F$.

3.2.2 Electrostatics

Suppose K is a timelike Killing field for the manifold, and define $X := g(K, K)$. Let

$$\mathfrak{E} := i_K F \tag{3.23}$$

$$\mathfrak{B} := i_K * F \tag{3.24}$$

$$\mathfrak{D} := i_K * M = l_x \mathfrak{E} + l_y \mathfrak{B} \tag{3.25}$$

$$\mathfrak{H} := i_K * *M = -i_K M = -l_y \mathfrak{E} + l_x \mathfrak{B} \tag{3.26}$$

where i_K denotes the interior product with the vector field K (i.e. $(i_K F)_\nu = K^\mu F_{\mu\nu}$). A general aether law will specify $\mathfrak{D}$ and $\mathfrak{B}$ as functions of $\mathfrak{E}$ and $\mathfrak{H}$ or vice versa.

Using Equations 3.23 and 3.24, we can rewrite the quantities x and y defined in Equations 3.20 and 3.21 above as

$$x = \frac{|\mathfrak{E}|^2 - |\mathfrak{B}|^2}{2X} \tag{3.27}$$

and

$$y = \frac{\mathfrak{E} \cdot \mathfrak{B}}{X} \tag{3.28}$$

It is clear that $y = 0$ whenever either $\mathfrak{E} = 0$ or $\mathfrak{B} = 0$. Furthermore, note that conservation of parity implies that $l(x, y) = l(x, -y)$, so that if we assume that l is differentiable in each argument with continuous first derivative, then $l_y(x, 0) = 0$. Hence we can say that whenever either $\mathfrak{E} = 0$ or $\mathfrak{B} = 0$, $l_y = 0$ also. Now using Equations 3.25 and 3.26 it is easy to see that in this case,

$$\mathfrak{D} = l_x \mathfrak{E} \tag{3.29}$$

and

$$\mathfrak{H} = l_x \mathfrak{B} \tag{3.30}$$

Thus $\mathfrak{E} = 0$ implies $\mathfrak{D} = 0$ and $\mathfrak{B} = 0$ implies $\mathfrak{H} = 0$.

Finally, the invariance of the equations under interchange of $\mathfrak{D}$ with $\mathfrak{B}$ and $\mathfrak{E}$ with $\mathfrak{H}$ implies that the system of equations in the case $\mathfrak{E} \equiv 0$ is formally the same as that in the case $\mathfrak{H} \equiv 0$. Thus in the case of electrostatic spacetimes ($\mathfrak{H} = 0$), we have $\mathfrak{B} = 0$, and hence $y = 0$. So we can say that in the case of electrostatics,

$$l(x, y) = l(x) \tag{3.31}$$

3.2.3 Stress-Energy Tensor

Finally, the stress-energy tensor T corresponding to l is a symmetric $\binom{0}{2}$ tensor field on $\mathcal{M}$ defined by

$$T_{\mu\nu} = 2\frac{\partial l}{\partial g^{\mu\nu}} - g_{\mu\nu} l \tag{3.32}$$

For the case of the electromagnetic Lagrangian as defined in Equation 3.19, this yields

$$T_{\mu\nu} = 2(l_x \frac{1}{2} F_{\mu\lambda} F_\nu{}^\lambda + l_y F_{\mu\lambda} * F_\nu{}^\lambda) - g_{\mu\nu} l = F_{\mu\lambda} M_\nu{}^\lambda - g_{\mu\nu} l \tag{3.33}$$

3.3 The Einstein-Maxwell System of PDEs

Combining Equations 3.10 and 3.18 gives the Einstein-Maxwell system of Partial Differential Equations:

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} & \text{(3.34a)} \\ dF = 0 & \text{(3.34b)} \\ dM = 0 & \text{(3.34c)} \end{cases}$$

which describes the geometry of a spacetime $(\mathcal{M}, g)$ endowed with an electromagnetic field.

In a static spherically symmetric spacetime $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, the metric can be written in the form

$$g_{\mu\nu} dx^\mu dx^\nu = -e^\xi dt^2 + e^\rho dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.35}$$

where (θ, ϕ) are spherical coordinates on the orbit spheres, and $\xi = \xi(r)$ and $\rho = \rho(r)$ are smooth functions that depend on the choice of aether law. It can be shown[12] that as a consequence of the Einstein-Maxwell system, $\rho = -\xi$, and we have

$$g_{\mu\nu} dx^\mu dx^\nu = -e^\xi dt^2 + e^{-\xi} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.36}$$

4 Point Defects in Electromagnetic Theory

The motivation for this project comes from the search for a consistent theory of electromagnetism that can describe the spacetime of a single point charge. The purpose of this section is to give a brief overview of the problem and the attempts that have been made to resolve it.

4.1 Linear Maxwell Theory

The simplest choice of aether law is Maxwell's $M = -*F$, corresponding to a Lagrangian with $l = x$. In this case, the Einstein-Maxwell system (Equations 3.34) becomes the "Einstein-Maxwell-Maxwell" system.

4.1.1 Reissner-Weyl-Nordström Spacetime

The unique spherically symmetric asymptotically flat solution of this system is known as the Reissner-Weyl-Nordström spacetime (RWN). Spherical symmetry implies that the electromagnetic field tensor can be written

$$F = E_r(r,t)dt \wedge dr + B_r(r,t)d\theta \wedge d\phi \tag{4.1}$$

This can be further simplified by noting that the nonexistence of magnetic monopoles implies that $B_r(r) = 0$, so we have

$$F = E_r(r,t)dt \wedge dr \tag{4.2}$$

Using Equation 3.34c with $M = -*F$ gives

$$d * F = \partial_\mu (E_r \sqrt{-g}) dx^\mu \wedge d\theta \wedge d\phi = 0 \tag{4.3}$$

with $\sqrt{-g} = r^2 \sin\theta$. Taking $\mu = 0$ (the t-component) implies that $E(r,t) = E(r)$, while taking $\mu = 1$ (the r-component) gives

$$\frac{\partial}{\partial r}(E_r(r) r^2) = 0 \tag{4.4}$$

$$\Rightarrow E_r(r) = -\frac{q_0}{r^2} \tag{4.5}$$

where q_0 is a constant, corresponding to the total charge of the spacetime. This corresponds to an electromagnetic potential

$$A = \psi(r)dt \tag{4.6}$$

where

$$\psi(r) = \frac{q_0}{r} \tag{4.7}$$

In linear Maxwell theory the stress-energy tensor for the electromagnetic field is given by

$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{4.8}$$

This is traceless ($T = 0$), so the Einstein Field Equation 3.15 becomes

$$R_{\mu\nu} = \kappa T_{\mu\nu} \tag{4.9}$$

We will now consider only the $\theta\theta$-component of this equation. It is easy to see that

$$T_{\theta\theta} = -\frac{1}{4}r^2 F_{\rho\sigma}F^{\rho\sigma} = \frac{q_0^2}{2r^2} \tag{4.10}$$

Furthermore, using Equations 2.15, 3.2 and 3.7 one can show that

$$R_{\theta\theta} = 1 - e^{\xi} - e^{\xi} r \frac{\partial \xi}{\partial r} \tag{4.11}$$

Substituting Equations 4.10 and 4.11 into Equation 4.9 and solving for e^{ξ} results in the following solution:[5]

$$e^{\xi} = \left(1 - \frac{2m_0}{r} + \frac{q_0^2}{r^2}\right) \tag{4.12}$$

where the constant m_0 is equal to the ADM mass, and was chosen so that the metric would become the Schwarzschild solution[6] in the limit $q_0 \to 0$.

The empirical charge-to-mass ratios of charged particles such as the electron and the proton (10^{18} and 10^{22} respectively) fit with the "superextremal regime", where $|q_0| > m_0$. It can be easily shown that in this case, the metric coefficient e^{ξ} is always positive, (t, r, θ, ϕ) is a global coordinate system for the manifold, and the only singularity present is on the timelike axis $r = 0$.

[5] Here the units have been chosen so that $\kappa = 1$.

[6] The Schwarzschild solution describes the spacetime outside a spherical mass with zero electric charge and angular momentum.

4.1.2 Problems with Infinite Self-Energy

The RWN solution has several difficulties associated with it, one of which is that the self-energy of the point charge is infinite. In general relativity, mass and energy are equivalent, so the total electrostatic energy will always make a contribution to the ADM mass of the spacetime. In the case of RWN, the total electrostatic energy carried by a time-slice is

$$\int_0^\infty |d\psi|^2 r^2 = \int_0^\infty \frac{q_0^2}{r^2} dr \tag{4.13}$$

which clearly makes an infinite contribution in the case of nonzero charge q_0.

The problem of the infinite self-energy of a point charge can be seen in classical Maxwell-Lorentz electrodynamics even before being coupled to gravity via the Einstein Field Equations.[10] Maxwell's electromagnetic field equations read:

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{s}) = -\nabla \times \mathbf{E}(t,\mathbf{s}) \tag{4.14}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}(t,\mathbf{s}) = \nabla \times \mathbf{H}(t,\mathbf{s}) - 4\pi\frac{1}{c}\mathbf{j}(t,\mathbf{s}) \tag{4.15}$$

$$\nabla \cdot \mathbf{B}(t,\mathbf{s}) = 0 \tag{4.16}$$

$$\nabla \cdot \mathbf{D}(t,\mathbf{s}) = 4\pi\rho(t,\mathbf{s}) \tag{4.17}$$

where c is the speed of light in a vacuum, $\mathbf{B}$ is the magnetic induction field, $\mathbf{E}$ is the electric field, $\mathbf{D}$ is the electric displacement field, $\mathbf{H}$ is the magnetic field, ρ is the electric charge density and $\mathbf{j}$ is the electric current vector-density. The latter two satisfy the local law of charge conservation,

$$\frac{\partial}{\partial t}\rho(t,\mathbf{s}) + \nabla \cdot \mathbf{j}(t,\mathbf{s}) = 0 \tag{4.18}$$

In addition, we have a "constitutive law" for matter-free space:

$$\mathbf{H}(t,\mathbf{s}) = \mathbf{B}(t,\mathbf{s}) \tag{4.19}$$

$$\mathbf{E}(t,\mathbf{s}) = \mathbf{D}(t,\mathbf{s}) \tag{4.20}$$

For a single electron with position vector $\mathbf{Q}(t) \in \mathbb{R}^3$, we have

$$\rho(t,\mathbf{s}) = -e\delta_{\mathbf{Q}(t)}(\mathbf{s}) \tag{4.21}$$

$$\mathbf{j}(t,\mathbf{s}) = -e\delta_{\mathbf{Q}(t)}(\mathbf{s})\dot{\mathbf{Q}}(t) \tag{4.22}$$

This assumes the electron to be a point particle with charge $-e$. For motion $t \to \mathbf{Q}(t)$, the Maxwell-Lorentz field equations are satisfied by

$$\mathbf{E}^{\text{ret}}(t,\mathbf{s}) = -e\frac{1}{(1-\mathbf{n}\cdot\dot{\mathbf{Q}}(\mathbf{t}))^3}\left(\frac{\mathbf{n}-\dot{\mathbf{Q}}/c}{\gamma^2 r^2}+\frac{\mathbf{n}\times[(\mathbf{n}-\dot{\mathbf{Q}}/c)\times\ddot{\mathbf{Q}}/c^2]}{r}\right)\bigg|_{\text{ret}} \tag{4.23}$$

$$\mathbf{B}^{\text{ret}}(t,\mathbf{s}) = \mathbf{n}|_{\text{ret}}\times\mathbf{E}^{\text{ret}}(t,\mathbf{s}) \tag{4.24}$$

where $\mathbf{n} = (\mathbf{s}-\mathbf{Q}(t))/r$, $r = |\mathbf{s}-\mathbf{Q}(t)|$ and $\gamma = (1-|\dot{\mathbf{Q}}(t)|^2/c^2)^{-1/2}$, and "ret" means that the function is to be evaluated at the retarded time t_{ret} defined by $c(t-t_{\text{ret}}) = |\mathbf{s}-\mathbf{Q}(t_{\text{ret}})|$. These are known as the "Liénard-Wiechert fields".

It is found that the Lorentz force

$$\mathbf{F}(t) = -e\left[\mathbf{E}(t,\mathbf{Q}(t))+\frac{1}{c}\dot{\mathbf{Q}}(t)\times\mathbf{B}(t,\mathbf{Q}(t))\right] \tag{4.25}$$

provides a highly accurate equation of motion for a test particle in an external field. However, it is clear that fundamentally there is no such thing as a "test charge". We must therefore consider the solutions to the field equations with the point charge as a source (i.e. Equations 4.23 and 4.24). However, these equations lead to a Lorentz "self-force" that is infinite in all directions (i.e. for any limit $\mathbf{s}\to\mathbf{Q}(t)$, the field magnitudes diverge to infinity while their limiting directions depend on how the limit is taken).

Attempts have been made to get past this problem by taking the limit $R\to 0$ over a sphere of radius R centred at $\mathbf{Q}(t)$. However, it can be shown[6] that such a method does not lead to a finite self-force, unless the acceleration vanishes at time t.

Renormalisation techniques[1], such as assuming the point charge to possess a "bare mass" of $-\infty$, have also been used in an attempt to remedy this problem, but these techniques are really more of an ad hoc procedure than a solution to the problem.

One further attempt at a solution was the "action-at-a-distance" theory, which asserts that a point charge is not directly affected by its own Liénard-Wiechert field.[13] However, this solution introduces other problems; in particular it is impossible to compute the accelerations of point charges at time t_0 without knowing the states of motion of all point electrons at infinitely many instances in the past and future.

4.1.3 Problems with the Naked Singularity

Another problem with the RWN solution is the presence of a strong naked singularity on the time axis, i.e. when $r = 0$. The Kretschmann scalar for the RWN metric is

$$\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48}{r^6}\left(m_0^2-\frac{2m_0q_0^2}{r}+\frac{7q_0^4}{6r^2}\right) \tag{4.26}$$

It is clear that the worst part of the singular behaviour at $r = 0$ arises from the contribution of the charge q_0.

It has been shown[11] that for $|q_0|/m_0 > 2$ naked singularities such as this one are linearly stable, but stability has not yet been proved in the nonlinear case (in fact, there is some confusion as to how this problem can even be formulated). Furthermore, the fact that the singularity is eternal means that the spacetime cannot arise as a solution of a classically-posed initial value problem.

4.2 Nonlinear Maxwell Theory

In order to overcome the problem of the infinite self-energy of a point charge, Max Born proposed to make the Maxwell equations nonlinear.[4] He suspected that these equations were valid only asymptotically, in the weak field limit. The new Lagrangian density needed to satisfy the following requirements:

(i) It must be Lorentz invariant, i.e. it must be unchanged under transformations of the form

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu \tag{4.27}$$

with $\det(\Lambda^\mu{}_\nu) = +1$.

(ii) It must be Weyl gauge invariant, i.e. it must be unchanged under transformations of the form

$$A^\mu \to A^\mu + \partial^\mu f \tag{4.28}$$

where f is a scalar field.

(iii) In the weak field limit it must reduce to the Maxwell-Maxwell Lagrangian:

$$L_0 = -\frac{1}{8\pi} F \wedge *F \tag{4.29}$$

(iv) It must yield finite field-energy solutions with point charge sources.

A one-parameter family of equations that satisfies these conditions is the set of Maxwell-Born-Infeld field equations. In this case, the Lagrangian density is[5]

$$L_\beta = *\frac{1}{4\pi\beta^4}\left[1 - \sqrt{1 - \beta^4 * (F \wedge *F) - \beta^8(*(F \wedge F))^2}\right] \tag{4.30}$$

where $\beta > 0$ is a parameter that gives a limit to the size of the field. Note that this is the only Lagrangian that both satisfies the above conditions, and has electromagnetic field equations that are completely linearly degenerate.[3]

In the case of electrostatics, $l(x, y) = l(x)$ (Equation 3.31), and we have

$$L_\beta = *\frac{1}{4\pi\beta^4}\left[1 - \sqrt{1 - \beta^4 * (F \wedge *F)}\right] \tag{4.31}$$

In the weak-field limit,

$$\begin{aligned} L_\beta &\approx *\frac{1}{4\pi\beta^4}\left[1 - (1 - \frac{1}{2}\beta^4 * (F \wedge *F))\right] && (4.32)\\ &= \frac{1}{4\pi\beta^4}\frac{\beta^4}{2} * *(F \wedge *F) && (4.33)\\ &= -\frac{1}{8\pi}F \wedge *F && (4.34) \end{aligned}$$

which is the Maxwell-Maxwell Lagrangian (the last line uses the fact that $** = -1$).

Near $r = 0$, the fields are stronger and this approximation is no longer valid, so one must use the completely nonlinear regime. In the electrostatic strong-field limit, this leads to an asymptotic Lagrangian

$$L_\beta \approx \frac{1}{4\pi\beta^4}\sqrt{-\beta^4 * (F \wedge *F)} \tag{4.35}$$

This Lagrangian leads to finite limits of the field strengths, and solves the problem of the infinite self-energy of a point charge. In addition, it is actually able to reduce the strength of the spacetime singularity that is present when the theory is coupled to gravity.[9] In this way it manages to deal with both of the problems discussed in Sections 4.1.2 and 4.1.3.

4.3 Static Spherically Symmetric Spacetime of a Single Point Charge

The solution of the Einstein-Maxwell system (Equations 3.34) depends upon the choice of aether law, which in turn depends upon the choice of Lagrangian. Here we will consider aether laws that are derivable from a Lagrangian, and that agree with that of Maxwell in the weak field limit.

The stress-energy tensor T, defined in Equation 3.33, is said to satisfy the Dominant Energy Condition if

(i) $T_{\mu\nu}Y^\mu Y^\nu \geq 0$ for every future-directed timelike vector Y.

(ii) The vector $-T^\mu{}_\nu Y^\nu$ is future-directed causal when Y is future-directed causal.

Essentially, this means that mass energy never flows faster than the speed of light, c.

It has been shown[12] that for any aether law that is derivable from a Lagrangian, satisfies the Dominant Energy Condition, agrees with Maxwell in the weak field limit and has a corresponding Hamiltonian satisfying certain growth conditions, there exists a unique electrostatic, spherically symmetric, asymptotically flat solution of the Einstein-Maxwell system with that aether law. Uniqueness is shown by a generalisation of Birkhoff's Theorem[2], which states that any spherically symmetric solution of the vacuum field equations is locally isometric to a region in Schwarzschild spacetime (and has been extended to prove uniqueness of the RWN solution in the case of spherical symmetry[8]).

This unique solution has a conical singularity on the time axis, which is the mildest possible singularity, and gives the spacetime the topology of $\mathbb{R}^4$ minus a line. It is found that the deficit angle of the conical singularity is proportional to the mass-to-charge ratio $m_0/|q_0|$, which is empirically small in the case of the electron and proton; hence in these cases the deviation from Minkowski spacetime is relatively small. In addition, the small mass-to-charge ratio means that, as in the case of RWN, there are no horizons of any kind and no trapped null geodesics. However, unlike the RWN solution, the naked singularity of this spacetime is gravitationally attractive.[12]

It can also be demonstrated that the electric field $\mathbf{E}$ is smooth everywhere except at $r = 0$, where there is a point defect at which the direction of the field is undefined. However, the total electrostatic energy of the field is finite and equal to the ADM mass of the spacetime, meaning that the mass of the point charge is entirely of electromagnetic origin.[12]

5 Removing Spherical Symmetry

It is known that in reality, particles such as electrons are not in fact spherically symmetric - they possess an intrinsic angular momentum or "spin". This section discusses an attempt to study the effects of an infinitesimal perturbation of the spherically symmetric solution on the spacetime of a point charge.

5.1 Infinitesimal Perturbation

As a first step to investigating how loss of spherical symmetry affects the spacetime, a one-parameter family of solutions to the Einstein-Maxwell system $(\tilde{g}, \tilde{F})$ was considered, such that

$$(\tilde{g}|_{\epsilon=0}, \tilde{F}|_{\epsilon=0}) = (g, F) \tag{5.1}$$

This family was expanded to first order in ϵ, giving

$$(\tilde{g}, \tilde{F}) \approx (g + \epsilon(\partial_\epsilon g)|_{\epsilon=0}, F + \epsilon(\partial_\epsilon F)|_{\epsilon=0}) \tag{5.2}$$

This is equivalent to perturbing each of g and F by an infinitesimal quantity, so that

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \epsilon X_{\mu\nu} \tag{5.3}$$

and

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} + \epsilon Y_{\mu\nu} \tag{5.4}$$

where $g_{\mu\nu}$ and $F_{\mu\nu}$ are the unique solutions to the static and spherically symmetric Einstein-Maxwell system for some Lagrangian (such as the Born-Infeld Lagrangian), as discussed in Section 4.3, and ϵ is an infinitesimally small quantity such that $\epsilon^2 = 0$. The tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$ are essentially the first derivatives of g and F with respect to the parameter ϵ, and were taken to have the following properties:

(i) *Time independence*: $X_{\mu\nu,t} = 0$ and $Y_{\mu\nu,t} = 0$

This condition follows from the fact that only stationary solutions are being studied here.

(ii) *Azimuthal symmetry*: $X_{\mu\nu,\phi} = 0$ and $Y_{\mu\nu,\phi} = 0$

This condition can be imposed without loss of generality by choosing the z-axis to coincide with the direction of angular momentum of the particle.

Furthermore, X is taken to be symmetric:

$$X_{\mu\nu} = X_{\nu\mu}$$

while Y is taken to be antisymmetric:

$$Y_{\mu\nu} = -Y_{\nu\mu}$$

These conditions follow from the symmetry properties of the metric tensor and the Faraday tensor respectively.

5.2 Linearising the Einstein-Maxwell System

The new values for $g_{\mu\nu}$ and $F_{\mu\nu}$ defined above were inserted into the Einstein-Maxwell system (Equations 3.34), neglecting terms of order ϵ^2. The result was a new set of equations for the quantities $X_{\mu\nu}$ and $Y_{\mu\nu}$. Of these, the only solutions that are physically meaningful are those defined modulo the Lie derivative of g and F (see Section 5.4.1).

5.2.1 Ricci Tensor and Ricci Scalar

The Christoffel symbols for the perturbed solution were calculated using Equation 2.15:

$$\Gamma^{\mu}{}_{\nu\lambda} = g'^{\mu\kappa}(g'_{\kappa\lambda,\nu} + g'_{\nu\kappa,\lambda} - g'_{\lambda\nu,\kappa})$$

where

$$g'^{\mu\nu} = g^{\mu\nu} - \epsilon X_{\sigma\kappa} g^{\sigma\mu} g^{\kappa\nu} \tag{5.5}$$

and

$$g_{\mu\nu} = \mathrm{diag}(-e^{\xi}, e^{-\xi}, r^2, r^2 \sin^2\theta) \tag{5.6}$$

i.e. the unperturbed spherically symmetric metric tensor.

These values for $\Gamma^{\mu}{}_{\nu\lambda}$ could then be used to calculate components of the Riemann tensor using Equation 3.2

$$R_{\mu}{}^{\nu}{}_{\lambda\rho} := \Gamma^{\nu}{}_{\lambda\mu,\rho} - \Gamma^{\nu}{}_{\rho\mu,\lambda} + \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\nu}{}_{\rho\sigma} - \Gamma^{\sigma}{}_{\rho\mu}\Gamma^{\nu}{}_{\lambda\sigma}$$

Finally, by contracting over the second and fourth indices, the Ricci tensor $R_{\mu\nu}$ could be constructed, as in Equation 3.7:

$$R_{\mu\nu} = R_{\mu}{}^{\lambda}{}_{\nu\lambda}$$

The values obtained for the Ricci tensor when spherical symmetry was removed are as follows:

$$
\begin{aligned}
R_{tt} = \frac{1}{2}e^{2\xi}\left(\frac{\partial^2\xi}{\partial r^2} + \left(\frac{\partial\xi}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial\xi}{\partial r}\right) + \epsilon\Bigg[& -\frac{1}{4}e^{\xi}\left(\frac{\partial\xi}{\partial r}\right)^2 X_{tt} \\
& + e^{\xi}\left(\frac{1}{4}\frac{\partial\xi}{\partial r} - \frac{1}{r}\right)X_{tt,r} - \frac{1}{2r^2}\frac{\cos\theta}{\sin\theta}X_{tt,\theta} - \frac{1}{2}e^{\xi}X_{tt,rr} - \frac{1}{2r^2}X_{tt,\theta\theta} \\
& - e^{\xi}\left(\frac{1}{2}\frac{\partial^2\xi}{\partial r^2} - \frac{1}{4}e^{2\xi}\left(\frac{\partial\xi}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial\xi}{\partial r}\right)X_{rr} + \frac{1}{2}e^{\xi}\frac{\partial\xi}{\partial r}\left(\frac{1}{2}e^{2\xi} - 1\right)X_{rr,r} \\
& + \frac{1}{2r^2}e^{2\xi}\frac{\partial\xi}{\partial r}\left(\frac{\cos\theta}{\sin\theta}X_{r\theta} - X_{r\theta,\theta} + \frac{1}{2}X_{\theta\theta,r} - \frac{2X_{\phi\phi}}{r\sin^2\theta} + \frac{X_{\phi\phi,r}}{\sin^2\theta}\right)\Bigg]
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
R_{tr} = R_{rt} = \epsilon\Bigg[& \frac{1}{2}e^{\xi}\left(\left(\frac{\partial\xi}{\partial r}\right)^2 + \frac{\partial^2\xi}{\partial r^2} - \frac{2}{r}\frac{\partial\xi}{\partial r}\right)X_{tr} + e^{\xi}\frac{\partial\xi}{\partial r}X_{tr,r} - \frac{1}{2r^2}\left(\frac{\cos\theta}{\sin\theta}X_{tr,\theta}\right. \\
& \left. - X_{tr,\theta\theta} - \frac{\cos\theta}{\sin\theta}\frac{\partial\xi}{\partial r}X_{t\theta} + \frac{\cos\theta}{\sin\theta}X_{t\theta,r} - \frac{\partial\xi}{\partial r}X_{t\theta,\theta} + X_{t\theta,r\theta}\right)\Bigg]
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
R_{t\theta} = R_{\theta t} = \epsilon\Bigg[& (e^{\xi} + e^{-\xi})\left(\frac{\partial\xi}{\partial r} - \frac{1}{2r}\right)X_{tr,\theta} - \frac{1}{2}e^{-\xi}X_{tr,r\theta} + \frac{1}{r}e^{\xi}\frac{\partial\xi}{\partial r}X_{t\theta} \\
& + \left(\frac{1}{2r}(e^{\xi} + e^{-\xi}) - \left(\frac{1}{2}e^{\xi} + e^{-\xi}\right)\frac{\partial\xi}{\partial r}\right)X_{t\theta,r} + \frac{1}{2}e^{-\xi}X_{t\theta,rr}\Bigg]
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
R_{t\phi} = R_{\phi t} = \epsilon\Bigg[& -\frac{1}{r}e^{\xi}\frac{\partial\xi}{\partial r}X_{t\phi} + \left(\frac{1}{4}(e^{\xi} - 2e^{-\xi})\frac{\partial\xi}{\partial r} + \frac{1}{2r}(e^{\xi} + e^{-\xi})\right)X_{t\phi,r} \\
& + \frac{1}{2}e^{-\xi}X_{t\phi,rr} + \frac{1}{2r^2}\frac{\cos\theta}{\sin\theta}X_{t\phi,\theta} - \frac{1}{2r^2}X_{t\phi,\theta\theta}\Bigg]
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
R_{rr} = &-\frac{1}{2}\left(\frac{\partial^2\xi}{\partial r^2}+\left(\frac{\partial\xi}{\partial r}\right)^2+\frac{2}{r}\frac{\partial\xi}{\partial r}\right)+\epsilon\Bigg[-e^{-\xi}\left(\frac{1}{2}\frac{\partial^2\xi}{\partial r^2}+\frac{5}{4}\left(\frac{\partial\xi}{\partial r}\right)^2\right)X_{tt}\\
&-\frac{1}{4}e^{-\xi}\frac{\partial\xi}{\partial r}X_{tt,r}+\frac{1}{2}e^{-\xi}X_{tt,rr}-\left(\frac{1}{4}e^{2\xi}\left(\frac{\partial\xi}{\partial r}\right)^2-\frac{e^{\xi}}{r}\frac{\partial\xi}{\partial r}\right)X_{rr}\\
&+e^{\xi}\left(\frac{1}{4}\frac{\partial\xi}{\partial r}+\frac{1}{r}\right)X_{rr,r}-\frac{1}{2r^2}\frac{\cos\theta}{\sin\theta}X_{rr,\theta}-\frac{1}{2r^2}X_{rr,\theta\theta}+\frac{1}{2r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial\xi}{\partial r}X_{r\theta}\\
&+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}X_{r\theta,r}+\frac{1}{2r^2}\frac{\partial\xi}{\partial r}X_{r\theta,\theta}+\frac{1}{r^2}X_{r\theta,r\theta}+\left(\frac{1}{4r^3}\frac{\partial\xi}{\partial r}-\frac{1}{r^4}\right)X_{\theta\theta}\\
&+\left(\frac{1}{r^3}-\frac{1}{4r^2}\frac{\partial\xi}{\partial r}\right)X_{\theta\theta,r}-\frac{1}{2r^2}X_{\theta\theta,rr}+\frac{1}{r^2\sin^2\theta}\Bigg(\left(\frac{1}{2r}\frac{\partial\xi}{\partial r}-\frac{1}{r^2}\right)X_{\phi\phi}\\
&+\left(\frac{1}{r}-\frac{1}{2}\frac{\partial\xi}{\partial r}\right)X_{\phi\phi,r}-\frac{1}{2}X_{\phi\phi,rr}\Bigg)\Bigg]
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
R_{r\theta} = R_{\theta r} = \epsilon\Bigg[&-\frac{1}{2}e^{-\xi}\left(\frac{1}{2}\frac{\partial\xi}{\partial r}+\frac{1}{r}\right)X_{tt,\theta}+\frac{1}{2}e^{-\xi}X_{tt,r\theta}+\frac{1}{2}e^{\xi}\left(\frac{5}{2}\frac{\partial\xi}{\partial r}+\frac{1}{r}\right)X_{rr,\theta}\\
&-\frac{1}{2}e^{\xi}\left(\frac{1}{r}\frac{\partial\xi}{\partial r}+\frac{3}{r^2}\right)X_{r\theta}+\frac{1}{2r^2}\frac{\cos\theta}{\sin\theta}\left(-\frac{2}{r}X_{\theta\theta}+X_{\theta\theta,r}-\frac{2}{r}X_{\phi\phi}+X_{\phi\phi,r}\right)\\
&+\frac{1}{2}\frac{1}{r^2\sin^2\theta}\left(\frac{2}{r}X_{\phi\phi,\theta}-X_{\phi\phi,r\theta}\right)\Bigg]
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
R_{r\phi} = R_{\phi r} = &\,\epsilon\left[e^{\xi}\left(\frac{1}{r}\frac{\partial\xi}{\partial r}\cos^2\theta-\frac{1}{2r}\frac{\partial\xi}{\partial r}\sin^2\theta+\frac{\sin^2\theta}{r^2}\right)X_{r\phi}\right]+\frac{e^{\xi}}{r}\cos^2\theta X_{r\phi,r}\\
&-\frac{1}{2r^2}\frac{\cos\theta}{\sin\theta}X_{r\phi,\theta}-\frac{1}{2r^2}X_{r\phi,\theta\theta}+\frac{1}{2r^2}\frac{\cos\theta}{\sin\theta}\left(\frac{2}{r}X_{\theta\phi}-X_{\theta\phi,r}\right)-\frac{1}{r^3}X_{\theta\phi,\theta}+\frac{1}{2r^2}X_{\theta\phi,r\theta}
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
R_{\theta\theta} = 1 - e^{\xi} - e^{\xi}\frac{\partial\xi}{\partial r}r + \epsilon\Bigg[&- \frac{1}{2}r\frac{\partial\xi}{\partial r}X_{tt} + \frac{1}{2}rX_{tt,r} + \left(3re^{3\xi}\frac{\partial\xi}{\partial r} + \frac{1}{2}re^{2\xi}\frac{\partial\xi}{\partial r} + e^{3\xi}\right)X_{rr} \\
&+ \left(e^{3\xi}r - \frac{1}{2}e^{2\xi}r\right)X_{rr,r} - \frac{1}{2}e^{\xi}X_{rr,\theta\theta} + \frac{e^{\xi}}{r}\frac{\cos\theta}{\sin\theta}X_{r\theta} + e^{\xi}\left(\frac{1}{r} + \frac{\partial\xi}{\partial r}\right)X_{r\theta,\theta} \\
&+ e^{\xi}X_{r\theta,r\theta} - \frac{1}{r^2}e^{\xi}X_{\theta\theta} - \frac{1}{2}e^{\xi}\left(\frac{\partial\xi}{\partial r} + \frac{1}{r}\right)X_{\theta\theta,r} + \frac{1}{2r^2}\frac{\cos\theta}{\sin\theta}X_{\theta\theta,\theta} \\
&- \frac{1}{2}e^{\xi}X_{\theta\theta,rr} + \frac{1}{r^2\sin^2\theta}\left(\frac{e^{\xi}}{\sin^2\theta} + 1\right)X_{\phi\phi} - \frac{e^{\xi}}{2r\sin^2\theta}X_{\phi\phi,r} \\
&- \frac{\cos\theta}{r^2\sin^3\theta}X_{\phi\phi,\theta} + \frac{1}{2r^2\sin^2\theta}X_{\phi\phi,\theta\theta}\Bigg]
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
R_{\theta\phi} = R_{\phi\theta} = \epsilon\Bigg[&- e^{\xi}\frac{\cos\theta}{\sin\theta}\left(\frac{\partial\xi}{\partial r} - \frac{1}{r} + \frac{\sin^2\theta}{r}\right)X_{r\phi} - e^{\xi}\frac{\cos\theta}{\sin\theta}X_{r\phi,r} + \frac{1}{2}e^{\xi}\frac{\partial\xi}{\partial r}X_{r\phi,\theta} \\
&+ \frac{1}{2}e^{\xi}X_{r\phi,r\theta} + \frac{1}{r^2}(1 - 2e^{\xi})X_{\theta\phi} + \frac{1}{2r}e^{\xi}X_{\theta\phi,r} - \frac{1}{2}e^{\xi}X_{\theta\phi,rr}\Bigg]
\end{aligned} \tag{5.15}
$$

$$
\begin{aligned}
R_{\phi\phi} = \sin^2\theta\left(1 - e^{\xi} - e^{\xi}\frac{\partial\xi}{\partial r}r\right) + \epsilon\Bigg[&- \frac{r}{2}\sin^2\theta\frac{\partial\xi}{\partial r}X_{tt} + \frac{r}{2}\sin^2\theta X_{tt,r} \\
&+ \frac{1}{2}\sin\theta\cos\theta e^{-\xi}X_{tt,\theta} - e^{2\xi}\sin^2\theta\left(1 + \frac{3}{2}r\frac{\partial\xi}{\partial r}\right)X_{rr} - \frac{3}{2}e^{2\xi}r\sin^2\theta X_{rr,r} \\
&- \frac{1}{2}e^{\xi}\sin\theta\cos\theta X_{rr,\theta} + e^{\xi}\sin\theta\cos\theta\left(\frac{1}{r} - \frac{\partial\xi}{\partial r}\right)X_{r\theta} - e^{\xi}\sin\theta\cos\theta X_{r\theta,r} \\
&+ \frac{e^{\xi}}{r}\sin^2\theta X_{r\theta,\theta} - \frac{1}{r^2}(\sin(2\theta) + \cos(2\theta) + e^{\xi}\sin^2\theta)X_{\theta\theta} - \frac{1}{2r}e^{\xi}\sin^2\theta X_{\theta\theta,r} \\
&+ \frac{1}{2r^2}\sin\theta\cos\theta X_{\theta\theta,\theta} - \frac{1}{r^2}\left(e^{\xi} - \frac{\cos^2\theta}{\sin^2\theta}\right)X_{\phi\phi} + \frac{1}{2}e^{\xi}\left(\frac{1}{r} + \frac{\partial\xi}{\partial r}\right)X_{\phi\phi,r} \\
&+ \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}X_{\phi\phi,\theta} + \frac{1}{2}e^{\xi}X_{\phi\phi,rr} + \frac{1}{4r^2}e^{\xi}\sin\theta\cos\theta X_{\phi\phi,\theta\theta}\Bigg]
\end{aligned} \tag{5.16}
$$

The Ricci scalar R was then constructed using Equation 3.8:

$$R = g^{\mu\nu}R_{\mu\nu}$$

with the following result:

$$
\begin{aligned}
R = \frac{2}{r^2} - e^{\xi}\left(\frac{\partial^2\xi}{\partial r^2} + \left(\frac{\partial\xi}{\partial r}\right)^2 - \frac{4}{r}\frac{\partial\xi}{\partial r} + \frac{2}{r}\right) + \epsilon\Bigg[&- \left(\frac{\partial^2\xi}{\partial r^2} + \frac{3}{2}\left(\frac{\partial\xi}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial\xi}{\partial r}\right) X_{tt} \\
&+ \left(\frac{2}{r} - \frac{1}{2}\frac{\partial\xi}{\partial r}\right) X_{tt,r} + \frac{e^{-\xi}}{r^2}\frac{\cos\theta}{\sin\theta}X_{tt,\theta} + X_{tt,rr} + \frac{e^{\xi}}{2r^2}X_{tt,\theta\theta} + \left(\frac{e^{2\xi}}{r}(e^{\xi}-1)\right. \\
&\left. + \frac{1}{2}(1+e^{2\xi})\frac{\partial^2\xi}{\partial r^2} + \frac{e^{2\xi}}{2}\left(1 - \frac{e^{\xi}}{2}\right)\left(\frac{\partial\xi}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial\xi}{\partial r}(3e^{3\xi} - e^{2\xi} - 1)\right) X_{rr} \\
&+ \left(\frac{1}{2}\frac{\partial\xi}{\partial r} + \frac{2e^{2\xi}}{r} + \frac{e^{3\xi}}{r}\right) X_{rr,r} - \frac{e^{\xi}}{r^2}\frac{\cos\theta}{\sin\theta}X_{rr,\theta} - \frac{e^{\xi}}{r^2}X_{rr,\theta\theta} + \frac{e^{\xi}}{r^3}\frac{\cos\theta}{\sin\theta}\left(\frac{2}{r} - \frac{\partial\xi}{\partial r}\right) X_{r\theta} \\
&+ \frac{e^{\xi}}{r}\left(\frac{3}{2}\frac{\partial\xi}{\partial r} + \frac{2}{r}\right) X_{r\theta,\theta} + \frac{2e^{\xi}}{r^2}X_{r\theta,r\theta} + \left(\frac{1}{r^4} - \frac{3}{4r^3}\frac{\partial\xi}{\partial r} - \frac{4e^{\xi}}{r^4}\right. \\
&\left. + \frac{\sin(2\theta)+\cos(2\theta)}{\sin^2\theta}\right) X_{\theta\theta} - \frac{e^{\xi}}{r^2}\frac{\partial\xi}{\partial r}X_{\theta\theta,r} + \frac{e^{\xi}}{2r^3\sin^3\theta}\frac{\partial\xi}{\partial r}X_{\phi\phi} \\
&+ \frac{e^{\xi}}{r^2\sin^2\theta}\left(\frac{1}{2}\frac{\partial\xi}{\partial r} + \frac{1}{r}\right) X_{\phi\phi,r} + \frac{1}{2r^4}\left(\frac{1}{\sin^2\theta} + \frac{\cos\theta}{2\sin\theta}\right) X_{\phi\phi,\theta\theta}\Bigg]
\end{aligned} \tag{5.17}
$$

5.2.2 Lagrangian

Recall Equations 3.19, 3.20 and 3.21, which demonstrate how the Lorentz-invariant gauge-invariant source-free Lagrangian depends only upon the electromagnetic invariants x and y, which in turn depend upon the Faraday tensor. The electromagnetic invariants are given by

$$x = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

and

$$y = \frac{1}{4}F_{\mu\nu} * F^{\mu\nu}$$

When perturbed using Equations 5.3 and 5.4, neglecting terms of order ϵ^2, they become

$$x = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\epsilon\left[F_{\mu\nu}g^{\mu\lambda}g^{\nu\kappa}Y_{\lambda\kappa} - F_{\mu\nu}g^{\mu\lambda}X^{\nu\kappa}F_{\lambda\kappa}\right] \tag{5.18}$$

and

$$
\begin{aligned}
y = \frac{1}{4}F_{\mu\nu} * F^{\mu\nu} - \epsilon\frac{1}{48r^2|\sin\theta|}\epsilon^{\mu\nu\lambda\kappa}\Bigg[&2F_{\mu\nu}Y_{\lambda\kappa} \\
&+ \frac{1}{2}\left(e^{-\xi}X_{tt} - e^{\xi}X_{rr} - \frac{X_{\theta\theta}}{r^2} - \frac{X_{\phi\phi}}{r^2\sin^2\theta}\right)F_{\mu\nu}F_{\lambda\kappa}\Bigg]
\end{aligned} \tag{5.19}
$$

The extra terms in the expression for y come from the fact that the Hodge star operator depends on the determinant of the metric, which varies as follows under the perturbation:

$$|g| = r^4 \sin^2\theta \left[1 + \epsilon\left(-e^{-\xi}X_{tt} + X_{rr}e^{\xi} + \frac{X_{\theta\theta}}{r^2} + \frac{X_{\phi\phi}}{r^2\sin^2\theta}\right)\right] \tag{5.20}$$

Using Equations 5.18 and 5.19, it is easy to find how the derivatives of l vary under the perturbation:

$$l_x \to l_x + \epsilon\left(l_{xx}\frac{\partial x}{\partial \epsilon} + l_{xy}\frac{\partial y}{\partial \epsilon}\right) \tag{5.21}$$

and

$$l_y \to l_y + \epsilon\left(l_{yx}\frac{\partial x}{\partial \epsilon} + l_{yy}\frac{\partial y}{\partial \epsilon}\right) \tag{5.22}$$

The variation in l itself can be found in the same way:

$$l \to l + l_x\frac{\partial x}{\partial \epsilon} + l_y\frac{\partial y}{\partial \epsilon} \tag{5.23}$$

5.2.3 Maxwell Tensor

Equation 3.22 gives an expression for $*M$ in terms of F, l_x and l_y. Since the effect of the perturbation on the latter three quantities is known (Equations 5.4, 5.21 and 5.22 respectively), this equation can be used to find the variation in $*M$. The result is as follows:

$$\begin{aligned}
(*M)_{\rho\sigma} = {}& l_x F_{\rho\sigma} + l_y\frac{1}{2}\epsilon_{\alpha\beta\rho\sigma}r^2|\sin\theta|g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta} + \epsilon\Bigg[\Bigg(l_{xx}\frac{1}{2}\Big(F_{\mu\nu}g^{\mu\nu}g^{\nu\kappa}Y_{\nu\kappa} \\
& - F_{\mu\nu}g^{\mu\lambda}X_{\nu\kappa}F_{\lambda\kappa}\Big) - l_{xy}\frac{1}{48r^2|\sin\theta|}\epsilon^{\mu\nu\lambda\kappa}\Big(\frac{1}{2}\Big(e^{-\xi}X_{tt} - e^{\xi}X_{rr} \\
& - \frac{X_{\theta\theta}}{r^2} - \frac{X_{\phi\phi}}{r^2\sin^2\theta}\Big)F_{\mu\nu}F_{\lambda\kappa} + 2F_{\mu\nu}Y_{\lambda\kappa}\Big)\Bigg)F_{\rho\sigma} + l_x Y_{\rho\sigma} \\
& + \frac{1}{2}r^2|\sin\theta|\epsilon_{\alpha\beta\rho\sigma}g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta}\Bigg(l_{yx}\frac{1}{2}\Big(F_{\mu\nu}g^{\mu\lambda}g^{\nu\kappa}Y_{\lambda\kappa} - F_{\mu\nu}g^{\mu\lambda}X_{\nu\kappa}F_{\lambda\kappa}\Big) \\
& - l_{yy}\frac{1}{48r^2|\sin\theta|}\epsilon^{\mu\nu\lambda\kappa}\Big(\frac{1}{2}\Big(e^{-\xi}X_{tt} - e^{\xi}X_{rr} - \frac{X_{\theta\theta}}{r^2} - \frac{X_{\phi\phi}}{r^2\sin^2\theta}\Big)F_{\mu\nu}F_{\lambda\kappa} \\
& + 2F_{\mu\nu}Y_{\lambda\kappa}\Big)\Bigg) + l_y\frac{1}{2}\epsilon_{\alpha\beta\rho\sigma}r^2|\sin\theta|\Big(\frac{1}{2}\Big(e^{-\xi}X_{tt} - e^{\xi}X_{rr} - \frac{X_{\theta\theta}}{r^2} \\
& - \frac{X_{\phi\phi}}{r^2\sin^2\theta}\Big)g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta} + 2X^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta} + g^{\alpha\gamma}g^{\beta\delta}Y_{\gamma\delta}\Big)\Bigg]
\end{aligned} \tag{5.24}$$

Using the fact that $** = -1$, the value of the Maxwell tensor can be found by applying the Hodge star operator to both sides of Equation 5.24:

$$M = -*(*M) \tag{5.25}$$

5.2.4 Stress-Energy Tensor

Finally, the variation of the stress-energy tensor can be found by applying Equations 5.24 (variation in $*M$) and 5.23 (variation in l) to the expression for the stress-energy tensor corresponding to the electromagnetic Lagrangian, given in Equation 3.33. The result is as follows:

$$\begin{aligned}
T_{\eta\rho} = F_{\eta\chi} g^{\chi\sigma} \left(l_x F_{\rho\sigma} + l_y \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} r^2 |\sin\theta| g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta} \right) - g_{\eta\rho} l \\
+ \epsilon \Bigg[\left(Y_{\eta\chi} g^{\chi\sigma} - F_{\eta\chi} X^{\chi\sigma} \right) \left(l_x F_{\rho\sigma} + l_y \epsilon_{\alpha\beta\rho\sigma} r^2 |\sin\theta| g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta} \right) \\
+ F_{\eta\chi} g^{\chi\sigma} \Bigg(\Bigg(l_{xx} \frac{1}{2} F_{\mu\nu} g^{\mu\lambda} (g^{\nu\kappa} Y_{\lambda\kappa} - X^{\nu\kappa} F_{\lambda\kappa}) - l_{xy} \frac{1}{48 r^2 |\sin\theta|} \epsilon^{\mu\nu\lambda\kappa} \Big(2 F_{\mu\nu} Y_{\lambda\kappa} \\
+ \frac{1}{2} \Big(e^{-\xi} X_{tt} - e^{\xi} X_{rr} - \frac{X_{\theta\theta}}{r^2} - \frac{X_{\phi\phi}}{r^2 \sin^2\theta} \Big) F_{\mu\nu} F_{\lambda\kappa} \Big) \Bigg) F_{\rho\sigma} + l_x Y_{\rho\sigma} \\
+ \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} r^2 |\sin\theta| g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta} \Big(l_{yx} \frac{1}{2} F_{\mu\nu} g^{\mu\lambda} (g^{\nu\kappa} Y \lambda\kappa - X^{\nu\kappa} F_{\lambda\kappa}) \\
- l_{yy} \frac{1}{48 r^2 |\sin\theta|} \epsilon^{\mu\nu\lambda\kappa} \Big(\frac{1}{2} \left(e^{-\xi} X_{tt} - e^{\xi} X_{rr} - \frac{X_{\theta\theta}}{r^2} - \frac{X_{\phi\phi}}{r^2 \sin^2\theta} \right) F_{\mu\nu} F_{\lambda\kappa} \\
+ 2 F_{\mu\nu} Y_{\lambda\kappa} \Big) \Big) + l_y \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} r^2 |\sin\theta| \Big(+ 2 X^{\alpha\gamma} g^{\beta\gamma} F_{\gamma\delta} + g^{\alpha\gamma} g^{\beta\delta} Y_{\gamma\delta} \\
+ \frac{1}{2} \left(e^{-\xi} X_{tt} - e^{\xi} X_{rr} - \frac{X_{\theta\theta}}{r^2} - \frac{X_{\phi\phi}}{r^2 \sin^2\theta} \right) g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta} \Big) \Bigg) + X_{\eta\rho} l \\
- g_{\eta\rho} \Bigg(l_x \frac{1}{2} F_{\mu\nu} g^{\mu\lambda} (g^{\nu\kappa} Y \lambda\kappa - X^{\nu\kappa} F_{\lambda\kappa}) - l_y \frac{1}{48 r^2 |\sin\theta|} \epsilon^{\mu\nu\lambda\kappa} \Big(2 F_{\mu\nu} F_{\lambda\kappa} \\
+ \frac{1}{2} \left(e^{-\xi} X_{tt} - e^{\xi} X_{rr} - \frac{X_{\theta\theta}}{r^2} - \frac{X_{\phi\phi}}{r^2 \sin^2\theta} \right) F_{\mu\nu} F_{\lambda\kappa} \Big) \Bigg) \Bigg]
\end{aligned} \tag{5.26}$$

5.3 The Perturbed Einstein-Maxwell System

Combining the expressions for $R_{\mu\nu}$, $g_{\mu\nu}$, R, $T_{\mu\nu}$, $F_{\mu\nu}$ and $M_{\mu\nu}$ derived in Section 5.2, one can rewrite the Einstein-Maxwell system (Equations 3.34) for an infinitesimal deviation from spherical symmetry. Knowing that the unperturbed quantities satisfy the Einstein-Maxwell equations, this leads to a new set of equations for the tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$.

The simplest example of this can be seen by inserting the perturbed Faraday tensor into Equation 3.34b. The exterior derivative of the Faraday tensor F can be found using Equation 2.24:

$$dF = d(F_{\mu\nu}dx^{\mu} \wedge dx^{\nu}) = \frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} dx^{\lambda} \wedge dx^{\mu} \wedge dx^{\nu} \tag{5.27}$$

If F is a solution of the Einstein-Maxwell system (Equations 3.34), then this quantity is equal to zero. When the perturbation is included, the equation becomes

$$d(F + \epsilon Y) = dF + \epsilon dY \tag{5.28}$$

$$= 0 + \epsilon dY \tag{5.29}$$

$$= \epsilon \frac{\partial Y_{\mu\nu}}{\partial x^{\lambda}} dx^{\lambda} \wedge dx^{\mu} \wedge dx^{\nu} \tag{5.30}$$

Hence, in order for the perturbed Faraday tensor to satisfy Equation 3.34b in the Einstein-Maxwell system, we must have

$$\frac{\partial Y_{\mu\nu}}{\partial x^{\lambda}} dx^{\lambda} \wedge dx^{\mu} \wedge dx^{\nu} = 0 \tag{5.31}$$

Because of the skew-symmetric property of the wedge product, this is in fact equivalent to saying that the tensor $Y_{\mu\nu}$ must satisfy

$$Y_{[\mu\nu,\lambda]} = 0 \tag{5.32}$$

This result is fairly trivial, given that Equation 3.34b is linear to begin with, but the example above does at least serve to demonstrate the method needed to find the equations for the perturbations X and Y.

Several checks were performed on the resulting equations to ensure that they made sense, both mathematically and physically. This included confirming that the dimensions of the Ricci tensor components and Ricci scalar were correct, and that the indices in tensor equations matched up. When this was done, it was possible to move on to further analysis.

5.4 Analysis of the Resulting Equations

5.4.1 Lie Derivatives of g and F

The only physically meaningful solutions to the linearised Einstein-Maxwell system are those defined modulo the Lie Derivatives of g and F.

If (g, F) is a solution to the Einstein-Maxwell system then so is (h_*g, h_*F) for any diffeomorphism $h(\epsilon)$. But we know that if there is a family of solutions $(\tilde{g}(\epsilon), \tilde{F}(\epsilon))$ such that

$$(\tilde{g}|_{\epsilon=0}, \tilde{F}|_{\epsilon=0}) = (g, F) \tag{5.33}$$

then

$$(X, Y) = ((\partial_\epsilon \tilde{g})|_{\epsilon=0}, (\partial_\epsilon \tilde{F})|_{\epsilon=0}) \tag{5.34}$$

satisfies the linearised Einstein-Maxwell system. Thus by setting $(\tilde{g}(\epsilon), \tilde{F}(\epsilon)) = (h_*g, h_*F)$, and recalling the definition of the Lie derivative given in Section 2.2.4, it is clear that

$$(X, Y) = ((\partial_\epsilon(h_*g))|_{\epsilon=0}, (\partial_\epsilon(h_*F))|_{\epsilon=0}) = (\mathcal{L}_X g, \mathcal{L}_X F) \tag{5.35}$$

is a solution to the linearised Einstein-Maxwell system. It is therefore evident that the only physically meaningful solutions to the system are those defined modulo these trivial solutions.

The components of the Lie derivative of a tensor are given by Equation 2.20. Finding the Lie derivatives of g and F in the direction of some arbitrary vector field Z gives

$$(\mathcal{L}_Z g)_{\mu\nu} = g_{\mu\nu,\lambda}Z^\lambda + g_{\lambda\nu}{Z^\lambda}_{,\mu} + g_\mu {Z^\lambda}_{,\nu} \tag{5.36}$$

and

$$(\mathcal{L}_Z F)_{\mu\nu} = F_{\mu\nu,\lambda}Z^\lambda + F_{\lambda\nu}{Z^\lambda}_{,\mu} + F_{\mu\lambda}{Z^\lambda}_{,\nu} \tag{5.37}$$

These two quantities together should satisfy the linearised Einstein-Maxwell system.

To confirm for example that $\mathcal{L}_Z F$ does in fact solve Equation 5.32, one can set

$$Y_{\mu\nu} = (\mathcal{L}_Z F)_{\mu\nu} \tag{5.38}$$

and take its derivative with respect to x^ρ:

$$Y_{\mu\nu,\rho} = F_{\mu\nu,\lambda\rho}Z^\lambda + F_{\mu\nu,\lambda}{Z^\lambda}_{,\rho} + F_{\lambda\nu,\rho}{Z^\lambda}_{,\mu} + F_{\lambda\nu}{Z^\lambda}_{,\mu\rho} + F_{\mu\lambda,\rho}{Z^\lambda}_{,\nu} + F_{\mu\lambda}{Z^\lambda}_{,\nu\rho} \tag{5.39}$$

Inserting this into Equation 5.32, and using the facts that $F_{\mu\nu}$ satisfies Equation 3.34b and $F[\mu\nu] = 0$, it is possible to show that $Y_{\mu\nu} = (\mathcal{L}_Z F)_{\mu\nu}$ is a solution to 3.34b in the perturbed system:

$$
\begin{aligned}
Y_{[\mu\nu,\rho]} = (F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\nu,\mu})_\lambda Z^\lambda + F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu})Z^\lambda{}_{,\rho} \\
+ (F_{\lambda\nu,\rho} + F_{\nu\rho,\lambda} + F_{\rho\lambda,\nu})Z^\lambda{}_{,\mu} + (F_{\mu\lambda,\rho} + F_{\lambda\rho,\mu} + F_{\rho\mu,\lambda})Z^\lambda{}_{,\nu} \\
+ (F_{\lambda\mu} + F_{\mu\lambda})Z_{,\mu\rho} + (F_{\mu\lambda} + F_{\lambda\nu})Z^\lambda{}_{,\nu\rho} + (F_{\rho\lambda} + F_{\lambda\rho})Z^\lambda{}_{,\mu\nu} \\
= 0 \quad (5.40)
\end{aligned}
$$

Similar calculations should indicate that the Lie derivatives of g and F also satisfy the perturbed equations corresponding to Equations 3.34a and 3.34c of the Einstein-Maxwell system, although the increased complexity of these equations in the absence of spherical symmetry makes this a much more difficult task.

5.4.2 Spherically Symmetric Solutions

As mentioned in Section 4.3, given an aether law that fits certain conditions, the spherically symmetric solution to the Einstein-Maxwell system is unique. This means that any spherically symmetric solutions to the linearised equations must be the trivial solution. To check this, one can remove all derivatives with respect to the angular coordinates and check that the only solutions to this simplified system are the trivial ones, i.e.

$$(X, Y) = (\mathcal{L}_Z g, \mathcal{L}_Z F) \qquad (5.41)$$

5.4.3 Difficulties with Analysis

Unfortunately, full analysis of the linearised equations was a near-impossible task. This was mostly due to the lengthiness and complexity of the expressions for $R_{\mu\nu}$, R, $M_{\mu\nu}$ and $T_{\mu\nu}$ when an infinitesimal perturbation had been added (see Equations 5.7- 5.17, 5.24 and 5.26), and meant that the linearisations of Equations 3.34a and 3.34c were so long that it was impossible to obtain any meaningful information from them in the time allotted for this project. Currently, alternative methods of linearisation are being considered, in the hopes that the resulting equations will be easier to work with.

5.5 An Alternative Method of Linearisation

One alternative method that could be used to linearise the Einstein-Maxwell system would be to return to the Lagrangian for the system and use it to derive the equations of motion for a one-parameter family of solutions. Then by varying the equations with respect to this parameter one can find the linearised equations for the perturbed system.

To demonstrate this method, we will show how it works for the simpler case of the "minimal surface problem", i.e. that of finding a surface of minimal area with a given boundary. Here we will consider the area as the Lagrangian of the problem, and denote it by L, where

$$L = (1 + g^{\alpha\beta} u_{;\alpha} u_{;\beta})^{1/2} \tag{5.42}$$

Then the problem amounts to finding a function u that minimises the integral

$$I = \int_\Omega L\eta = \int_\Omega (1 + g^{\alpha\beta} u_{;\alpha} u_{;\beta})^{1/2}\,\eta \tag{5.43}$$

where η is the volume form. First, to find the Euler-Lagrange equations, we vary I with respect to a parameter s:

$$\frac{\partial I}{\partial s} = \int_\Omega \frac{\partial L}{\partial s}\,\eta \tag{5.44}$$

where

$$\frac{\partial L}{\partial s} = \frac{1}{2} L^{-1} \left(g^{\alpha\beta} \frac{\partial u_{;\alpha}}{\partial s} u_{;\beta} + g^{\alpha\beta} u_{;\alpha} \frac{\partial u_{;\beta}}{\partial s} \right) = L^{-1} g^{\alpha\beta} u_{;\alpha} \frac{\partial u_{;\beta}}{\partial s} \tag{5.45}$$

I is stationary if and only if

$$\int_\Omega L^{-1} g^{\alpha\beta} u_{;\alpha} \frac{\partial u_{;\beta}}{\partial s}\,\eta = 0 \tag{5.46}$$

for all u such that $\frac{\partial u}{\partial s}|_{\partial\Omega} = 0$, or equivalently, that

$$(L^{-1} g^{\alpha\beta} u_{;\alpha})_{;\beta} = 0 \tag{5.47}$$

pointwise (using integration by parts and neglecting boundary terms involving $\frac{\partial u}{\partial s}$). These are the Euler-Lagrange equations for the system.

Now, we will assume that we have a solution $\tilde{u}$ from the one-parameter family of solutions[7] and vary the system with respect to this second parameter, ϵ. We must therefore calculate

$$\frac{\partial^2 I}{\partial\epsilon\partial s} = \int_\Omega \frac{\partial^2 L}{\partial\epsilon\partial s} \tag{5.48}$$

[7]Note that every equation satisfied by u is also satisfied by $\tilde{u}$.

where

$$\begin{aligned}
\frac{\partial^2 L}{\partial\epsilon\partial s} &= -L^{-2}\frac{\partial L}{\partial\epsilon}g^{\alpha\beta}\tilde{u}_{;\alpha}\frac{\partial\tilde{u}_{;b}}{\partial s} + L^{-1}g^{\alpha\beta}\frac{\partial\tilde{u}_{;\alpha}}{\partial\epsilon}\frac{\partial\tilde{u}_{;\beta}}{\partial s} + L^{-1}g^{\alpha\beta}\tilde{u}_{;\alpha}\frac{\partial^2\tilde{u}_{;\beta}}{\partial\epsilon\partial s} \\
&= -L^3 g^{\gamma\delta}\tilde{u}_{;\gamma}\frac{\partial\tilde{u}_{;\delta}}{\partial\epsilon}g^{\alpha\beta}\tilde{u}_{;\alpha}\frac{\partial\tilde{u}_{;\beta}}{\partial s} + L^{-3}(1+g^{\gamma\delta}\tilde{u}_{;\gamma}\tilde{u}_{;\delta})g^{\alpha\beta}\frac{\partial\tilde{u}_{;\alpha}}{\partial\epsilon}\frac{\partial\tilde{u}_{;\beta}}{\partial s} + L^{-1}g^{\alpha\beta}\tilde{u}_{;\alpha}\frac{\partial^2\tilde{u}_{;\beta}}{\partial\epsilon\partial s} \\
&= L^{-3}(g^{\alpha\beta} + g^{\alpha\beta}g^{\gamma\delta}\tilde{u}_{;\gamma}\tilde{u}_{;\delta} - g^{\alpha\gamma}g^{\beta\delta}\tilde{u}_{;\gamma}\tilde{u}_{;\delta})\frac{\partial\tilde{u}_{;\alpha}}{\partial\epsilon}\frac{\partial\tilde{u}_{;\beta}}{\partial s} + L^{-1}g^{\alpha\beta}\tilde{u}_{;\alpha}\frac{\partial^2\tilde{u}_{;\beta}}{\partial\epsilon\partial s} \\
&\equiv L^{-3}h^{\alpha\beta}\frac{\partial\tilde{u}_{;\alpha}}{\partial\epsilon}\frac{\partial\tilde{u}_{;\beta}}{\partial s} + L^{-1}g^{\alpha\beta}\tilde{u}_{;\alpha}\frac{\partial^2\tilde{u}_{;\beta}}{\partial\epsilon\partial s} \qquad (5.49)
\end{aligned}$$

This time we want the integral in Equation 5.48 to be equal to 0 whenever $\frac{\partial\tilde{u}}{\partial s}$ and $\frac{\partial^2\tilde{u}}{\partial\epsilon\partial s}$ vanish on the boundary. Therefore, using a similar integration by parts trick as before, we have

$$\int_\Omega \frac{\partial^2 L}{\partial\epsilon\partial s}\eta = -\int_\Omega \left(h^{\alpha\beta}\frac{\partial\tilde{u}_{;\alpha}}{\partial\epsilon}_{;\beta}\frac{\partial\tilde{u}}{\partial s}\right)\eta - \int_\Omega (L^{-1}g^{\alpha\beta}\tilde{u}_{;\alpha})_{;\beta}\frac{\partial^2\tilde{u}}{\partial\epsilon\partial s}\eta \qquad (5.50)$$

The second term here is zero by the Euler-Lagrange equations (Equation 5.47), so we merely require that

$$\int_\Omega \left(h^{\alpha\beta}\frac{\partial\tilde{u}_{;\alpha}}{\partial\epsilon}\right)_{;\beta}\frac{\partial\tilde{u}}{\partial s}\eta = 0 \qquad (5.51)$$

for all $\tilde{u}$ such that $\frac{\partial\tilde{u}}{\partial s}|_{\partial\Omega} = 0$, or equivalently, that

$$\left(h^{\alpha\beta}\frac{\partial\tilde{u}_{;\alpha}}{\partial\epsilon}\right)_{;\beta} = 0 \qquad (5.52)$$

pointwise.

The equations found using this method should be equivalent to those found using the methods described in Sections 5.1, 5.2 and 5.3. To solve the minimal surface problem using an infinitesimal perturbation, one would substitute $u' = u + \epsilon v$ for u in the minimal surface equation (Equation 5.47) and keep only terms that are linear in ϵ. This would give an equation for v, which should be equivalent to that found for $\frac{\partial\tilde{u}_{;\alpha}}{\partial\epsilon}$ using the second method (Equation 5.52).

The quantity v is equivalent to the quantities X and Y in the problem of linearising the Einstein-Maxwell system. The main difference is that these equations are derived using a completely general background solution, which can be substituted in at the end, while the first method made use of the particular background value for g right from the start. It is possible that by doing the calculation in this second way, the resulting equations could be easier to work with than those found in Section 5.2 above.

6 Conclusions and Further Study

In this project, an attempt was made to examine perturbations of the spherically symmetric solutions of the Einstein-Maxwell system of PDEs with a nonlinear aether law. This was done by adding infinitesimally small perturbations $\epsilon X_{\mu\nu}$ and $\epsilon Y_{\mu\nu}$ to the metric $g_{\mu\nu}$ and Faraday tensor $F_{\mu\nu}$ respectively, inserting them into the Einstein-Maxwell system, and neglecting terms of order ϵ^2. The result was a new set of equations for the quantities $X_{\mu\nu}$ and $Y_{\mu\nu}$. As discussed in Section 5.4.3, the expressions obtained for $R_{\mu\nu}$, R, $M_{\mu\nu}$ and $T_{\mu\nu}$ were very lengthy and complicated (see Equations 5.7-5.17, 5.24 and 5.26). This meant that a full analysis of the resulting linearised system of equations was near-impossible given the time allotted for this project. Currently, alternative methods of linearising the Einstein-Maxwell system are being considered, such as that discussed in Section 5.5. In this method, the known "background" solution is not substituted in until the end, meaning that the form of the equations is more compact for the majority of the calculation. It is hoped that final form of the equations will be simpler in this case as well.

Future work will involve either trying to carry out the calculation using this (or another) alternative method, or else attempting to further analyse the equations found using the original method. The ultimate goal is to find the solutions for the next spherical harmonic, and to investigate which of the three outcomes of the calculation mentioned in the Introduction (i.e. that a solution exists (i) for each mass, charge and angular momentum, (ii) only for some values of angular momentum or (iii) for no nonzero values of angular momentum) is correct in the case of the Born-Infeld Lagrangian. The result will give some insight into the suitability of nonlinear Maxwell theory with Born-Infeld Lagrangian as a description of real-world charged particles such as the electron and proton.

7 Acknowledgements

I would like to thank my project supervisor Prof John Stalker for introducing me to the field of nonlinear electromagnetism in general relativity, and for his invaluable guidance throughout the project.

I would also like to thank Aoibheann Brady for her help with LaTeX and Caoimhín Laoide-Kemp for all the help with proof-reading (and for the delicious cookies too!).

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