Nonlinear Electromagnetism in General Relativity

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Motivation

- Need a **consistent** theory of electromagnetism that can describe the spacetime of a **single point charge**
- Linear theories lead to several problems
  $\longrightarrow$ Try nonlinear electromagnetism
- Focus up to now: spherically symmetric solutions
- In this project: study how removal of spherical symmetry affects the spacetime

Overview

1. Einstein-Maxwell System
2. Linear Maxwell Theory
3. Nonlinear Maxwell Theory
4. Removing Spherical Symmetry
5. Conclusions

Einstein's Field Equations

- Spacetime $(\mathcal{M}, g)$ with corresponding Riemann curvature tensor $R_{\mu}{}^{\nu}{}_{\lambda\kappa}$

Einstein's Field Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$$

- Ricci tensor: $R_{\mu\nu} = R_{\mu}{}^{\lambda}{}_{\nu\lambda}$
- Ricci scalar: $R = g^{\mu\nu} R_{\mu\nu}$
- Stress-energy tensor $T_{\mu\nu}$: derived from the Lagrangian

Source-Free Maxwell Equations

- Faraday tensor: $F = dA$, where A is a critical point of

$$S[a] = \int L_{em}(a, da)$$

- Maxwell tensor:

$$M = \left.\frac{\partial L_{em}}{\partial f}\right|_{a=A, f=F}$$

Source-Free Maxwell Equations

$$\begin{cases} dF = 0 \\ dM = 0 \end{cases}$$

The Einstein-Maxwell System of PDEs

Einstein's Field Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$$

Source-Free Maxwell Equations

$$\begin{cases} dF = 0 \\ dM = 0 \end{cases}$$

$$\Downarrow \qquad \Downarrow$$

Einstein-Maxwell System

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \\ dF = 0 \\ dM = 0 \end{cases}$$

Aether Laws

- Aether Law: Choice of relationship between M and F
- Depends on choice of Lagrangian

$$M = \left.\frac{\partial L_{em}}{\partial f}\right|_{a=A, f=F}$$

- Linear Maxwell theory:

$$L_{em} = -\frac{1}{8\pi} F \wedge *F$$

$$\Rightarrow M = - * F$$

Reissner-Weyl-Nordström Spacetime

- Linear Maxwell aether law:

$$M = - * F$$

- Static spherically symmetric case:

$$g_{\mu\nu} dx^\mu dx^\nu = -e^\xi dt^2 + e^{-\xi} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

- Unique asymptotically flat solution:

$$e^\xi = \left(1 - \frac{2m_0}{r} + \frac{q_0^2}{r^2}\right) \qquad F = -\frac{q_0}{r^2} dt \wedge dr$$

Problems with Linear Maxwell Theory

- Infinite contribution to ADM mass:

$$\text{Electrostatic energy} = \int_0^\infty \frac{q_0^2}{r^2} dr \longrightarrow \infty$$

- Strong naked singularity on time axis:

$$\mathcal{K} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{48}{r^6} \left(m_0^2 - \frac{2 m_0 q_0^2}{r} + \frac{7 q_0^4}{6 r^2} \right)$$

Requirements for Nonlinear Lagrangian Density

- Lorentz invariant

$$X^{\mu} \rightarrow \Lambda^{\mu}{}_{\nu} X^{\nu}, \quad \det(\lambda^{\mu}{}_{\nu}) = +1$$

- Weyl gauge invariant:

$$A^{\mu} \rightarrow A^{\mu} + \partial^{\mu} f, \quad f \text{ is a scalar field}$$

- In weak field limit, must reduce to

$$L_{Maxwell} = -\frac{1}{8\pi} F \wedge *F$$

- Finite field-energy solutions with point charge sources

Maxwell-Born-Infeld Lagrangian

Maxwell-Born-Infeld Lagrangian

$$L_\beta = *\frac{1}{4\pi\beta^4}\left[1 - \sqrt{1 - \beta^4 * (F \wedge *F) - \beta^8(*(F \wedge F))^2}\right]$$

- Finite limits of field strengths
- Reduces strength of spacetime singularity
- Reduces to Maxwell Lagrangian in weak field limit

Spherically Symmetric Spacetime of a Single Point Charge

Theorem (A. Shadi Tahvildar-Zadeh, 2011)

For any aether law that

- *is derivable from a Lagrangian*
- *satisfies the Dominant Energy Condition*
- *agrees with Maxwell in the weak field limit*
- *satisfies certain growth conditions*

$\exists$ *a unique electrostatic, spherically symmetric, asymptotically flat solution of the Einstein-Maxwell system with*

- *finite electric field strength*
- *mild conical singularity on time axis*

Removing Spherical Symmetry

Possible Outcomes:

1. $\exists$ a solution for each mass, charge and angular momentum
2. $\exists$ a solution only for some values of angular momentum
3. $\nexists$ solutions for nonzero angular momentum

Infinitesimal Perturbation

- Infinitesimal perturbation added to spherically symmetric solution (g, F):

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \epsilon X_{\mu\nu}$$

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} + \epsilon Y_{\mu\nu}$$

- Re-insert into Einstein-Maxwell system, neglecting terms of order ϵ^2

$\longrightarrow$ Linearised Einstein-Maxwell System for X and Y

Lie Derivatives of F and g

- Linearised system trivially satisfied by Lie derivatives of (g, F):

$$(X, Y) = (\mathcal{L}_Z g, \mathcal{L}_Z F)$$

- Physically meaningful solutions defined modulo these trivial solutions

Problems Encountered

- Linearised equations were lengthier and more complex than originally expected
- Possible alternative method:
  - Use Lagrangian to derive EOM for a one-parameter family
  - Vary wrt this parameter to find the linearised equations

Conclusions and Further Study

- Nonlinear electromagnetism solves problems of infinite self-energy
- Important to investigate non-spherically symmetric solutions
- Equations more complicated than expected; alternative methods currently being considered
- Ultimate goal: find solutions for next spherical harmonic